THE NUMBER *e*

Recall the special number π. It is the ratio of the circumference of any circle to its diameter, and we learned that it's irrational (an infinite, nonrepeating decimal). There's another extremely important irrational number and it's the topic of this chapter. It will be defined in terms of a limit, and will be the basis of applications in banking, biology, and radioactivity. Moreover, in Calculus it's absolutely the most important base for exponential functions.

❑ NEW LIMIT NOTATION

Let's review an example of limits. Consider the function

$$f(x) = \frac{1}{x}.$$

We know that as x grows larger and larger, $f(x)$ gets closer and closer to 0. We usually write:

$$\text{As } x \to \infty,\ f(x) \to 0.$$

In our new notation, we would write this fact as

$$\lim_{x \to \infty} f(x) = 0.$$

In general, instead of writing

$$\text{As } x \to a, f(x) \to L,$$

we write

$$\lim_{x \to a} f(x) = L.$$

These two statements mean exactly the same thing.

Homework

1. If $h(x) = x^2$, find $\lim_{x \to 3} h(x)$.

2. If $f(x) = \dfrac{2x+1}{x-3}$, find $\lim_{x \to \infty} f(x)$. Hint: Consider the graph of f.

3. If $E(x) = \dfrac{9}{1+x^2}$, find $\lim_{x \to 0} E(x)$.

❑ *Earning Interest at the Bank*

In **simple interest** at the bank, the interest you earn on your investment is calculated just once at the end of the year. Suppose you place $70 in the bank at 10% interest per year. At the end of the year you will have earned $7 in interest, making your current balance $77.

Now let's consider **compound interest**. As an example, we'll divide the year into four compounding periods (each one called a *quarter*), and we'll pretend we have $1 invested at 100% annual interest. At the end of each quarter, you earn interest in the amount of 25% (100% divided into 4 equal parts) of your current balance. This means that from now on, your interest itself is earning interest.

Start: $1

1st Quarter: previous balance + 25% of the previous balance

$= \$1 + 25\% \text{ of } \$1$

$= \$1 + \$\frac{1}{4}$ (balance = $1.25 after 1 quarter)

2nd Quarter: previous balance + 25% of the previous balance

$= (\$1 + \$\frac{1}{4}) + 25\% \text{ of } (\$1 + \$\frac{1}{4})$

$= (\$1 + \$\frac{1}{4}) + \frac{1}{4}(\$1 + \$\frac{1}{4})$

$= (\$1 + \$\frac{1}{4})(\$1 + \$\frac{1}{4})$ (factor out $1 + \frac{1}{4}$)

$= (\$1 + \$\frac{1}{4})^2$ (balance = \$1.56 after 2 quarters)

<u>3rd Quarter:</u> previous balance + 25% of the previous balance

$= (\$1 + \$\frac{1}{4})^2 + 25\% \text{ of } (\$1 + \$\frac{1}{4})^2$

$= (\$1 + \$\frac{1}{4})^2 + \frac{1}{4}(\$1 + \$\frac{1}{4})^2$

$= (\$1 + \$\frac{1}{4})^2(\$1 + \$\frac{1}{4})$ (factor out $(1 + \frac{1}{4})^2$)

$= (\$1 + \$\frac{1}{4})^3$ (balance = \$1.95 after 3 quarters)

<u>4th Quarter:</u> previous balance + 25% of the previous balance

$= (\$1 + \$\frac{1}{4})^3 + 25\% \text{ of } (\$1 + \$\frac{1}{4})^3$

$= (\$1 + \$\frac{1}{4})^3 + \frac{1}{4}(\$1 + \$\frac{1}{4})^3$

$= (\$1 + \$\frac{1}{4})^3(\$1 + \$\frac{1}{4})$ (factor out $(1 + \frac{1}{4})^3$)

$= (\$1 + \$\frac{1}{4})^4$ (balance = \$2.44 after 1 full year)

<u>Note:</u> To compare simple interest with compound interest, \$1 invested for one year at 100% <u>simple</u> interest would yield \$1 in interest, for a final balance of **\$2**. Using <u>compound</u> interest, we see that the last calculation in the 4th quarter above is

$$(\$1 + \$\tfrac{1}{4})^4 = \$1.25^4 = \mathbf{\$2.44}$$

To summarize, investing \$1 at 100% annual interest compounded four times a year yields a final balance, after one year in the bank, of

$$(\$1 + \$\tfrac{1}{4})^4$$

From this example we can generalize as follows: Investing \$1 at 100% annual interest compounded n times a year yields a final balance of

$$\left(\$1 + \$\frac{1}{n}\right)^n$$

at the end of one year. For instance, investing \$1 at 100% annual interest compounded 360 times a year would give you a balance of

$$\left(1 + \frac{1}{360}\right)^{360} = (1.002777778)^{360} = \$2.7145$$

Summarizing,

> Investing \$1 at 100% annual interest for one year, compounded *n* times a year, yields a final balance of
>
> $$\left(1 + \frac{1}{n}\right)^n$$

Homework

4. For each problem, use the formula $\left(1 + \frac{1}{n}\right)^n$ to find the balance at the end of one year if the original investment was \$1, earning 100% annual interest, and assuming the given number of compounding periods:

   a. 1 b. 12 c. 365 d. 1,000 e. 5,000,000

❑ *THE MOST IMPORTANT EXPONENTIAL BASE OF ALL !*

Now, suppose that we consider compounding so frequently that it occurs at every instant of time. This is called ***continuous compounding***. What will your account balance be after one year of continuous compounding? It would come from the formula

> Whatever every instant of time means; this drove the Greeks bonkers, and it wasn't until the 17th century that Newton and Leibniz made some sense of this when they invented Calculus.]

$$\left(1+\frac{1}{n}\right)^n$$

with n replaced by ∞: $\left(1+\frac{1}{\infty}\right)^\infty$. But this would be meaningless. So we use the new limit notation introduced a few pages back. We keep the n's in the formula, and then specify that n should approach infinity; that is, become infinitely large:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It's hard to see this now, but the limit expression above is actually a specific real number. In fact, this number is one of the most important numbers in math, science, statistics, and business. It is given the name "***e***", perhaps due to the word "*e*xponential," or perhaps due to its creator Leonard *E*uler.

Though ***e*** is defined as a limit, a decimal approximation of ***e*** will be discussed in the homework. In summary,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Homework

5. Define e in two ways: as a limit, and as the result of a certain investment.

6. The number e is irrational, although I have no idea how to prove it. Nevertheless, explain exactly what it means for e to be irrational.

7. a. Use the definition of e with a value of $n = 10{,}000{,}000$ to approximate the value of e.

   b. Use the e^x button on your calculator to approximate e. Hint: $e = e^1$.

   c. Which approximation do you think is more accurate?

8. Use your calculator to approximate each of the following:

   a. $e^{3.7}$ b. $\dfrac{1}{e}$ c. $\sqrt[7]{e}$ d. $\dfrac{e^2 - \sqrt{e}}{\sqrt[3]{4e+1}}$

9. Some books define ***e*** as $\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^n$. Prove that this is the same as our definition.

❑ *The Laws of Exponents*

All the laws for exponents with which we're familiar work for exponential expressions like 2^x and e^x just as well as they did for x^2. Here are the six main rules. Assume a and b are positive constants.

$$a^{-x} = \frac{1}{a^x} \qquad a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Homework

10. Simplify each expression:

a. $2^x 2^y$ b. e^0 c. e^{-5} d. $\dfrac{e^5}{e^2}$ e. $\dfrac{e^x}{e^y}$

f. $(e^x)^e$ g. $(10e)^x$ h. $\left(\dfrac{e}{2}\right)^n$ i. $(e^\pi)^{10x}$ j. $2^3 2^0$

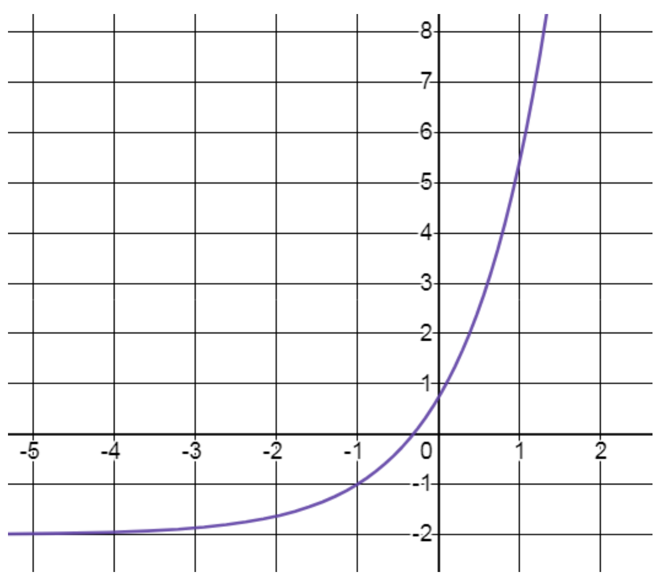
❑ *GRAPHING BASE-e EXPONENTIAL FUNCTIONS*

Since *e* is a positive real number not equal to 1 (in fact, it's greater than 1), it's allowed to be the base of an exponential function. In fact, the base-*e* exponential function, $\boldsymbol{y = e^x}$, is one of the most important exponential functions in business, biology, chemistry, ecology, statistics, physics, and electronics. Let's graph one of these.

EXAMPLE 1: **Graph:** $\boldsymbol{y = e^{x+1} - 2}$

Solution: Use your calculator to verify that each of the following ordered pairs is on the graph of the function:

(–4, –1.95) (–3, –1.86) (–2, –1.63) (–1, –1) (0, 0.72) (1, 5.39)



Exponential Growth

The domain of the function is $\mathbb{R}$. To determine the range, we need to verify what the graph appears to show: that the line $y = -2$ is a horizontal asymptote. Let x be a large negative number, for instance $x = -20$. Then the y-value is

$$y = e^{-20+1} - 2 = e^{-19} - 2 = .000000006 - 2 = -1.999999994$$

which is extremely close to (and slightly above) –2. We can be pretty confident that the horizontal asymptote is $y = -2$. From this fact, we can conclude that the range of the function is all real numbers greater than –2: **(–2, ∞)**.

For our last bit of analysis, we summarize the behavior of the function at extreme values of x by writing the limits

$$\lim_{x\to\infty}(e^{x+1} - 2) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty}(e^{x+1} - 2) = -2$$

The first of these two limits indicates that the graph is increasing; that is, as you move from left to right along the graph, the graph grows taller and taller (and never levels off). We can call this **exponential growth**. In our following example, we change one little (but critical) part of the equation (which part?) and we get a curve which represents **exponential decay**.

EXAMPLE 2: **Graph:** $y = e^{-x} + 1$

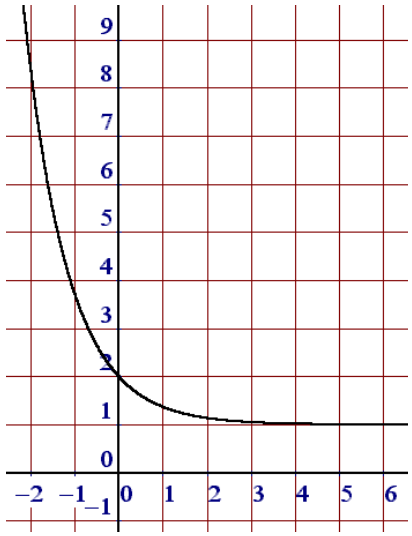
Solution: Let's go for the y-intercept first. Letting $x = 0$ produces $y = e^{-0} + 1 = e^{0} + 1 = 1 + 1 = 2$. The y-intercept is therefore **(0, 2)**. Use your calculator to verify some other points on the graph:

(–3, 21.09) (–2, 8.39) (–1, 3.72) (1, 1.37) (2, 1.14) (3, 1.05)

To emphasize the limits as x approaches ∞ or –∞, let's calculate two more points:

(–10, 22027.47) (10, 1.000045)

It appears that as $x \to -\infty$, $y \to \infty$. Also, as $x \to \infty$, $y \to 1$



Exponential Decay

The domain of the function is $\mathbb{R}$, simply because any real number x can be used in the formula without a problem.

The range is all real numbers greater than 1: **(1, ∞)**

From the graph and the points we calculated, we see that there's a horizontal asymptote at $\boldsymbol{y = 1}$.

Homework

Graph each of the following exponential functions:

11. $y = e^x$

12. $f(x) = e^{-x}$

13. $g(x) = e^2 + 2$

14. $h(x) = e^{x-2}$

15. $f(x) = e^x - 1$

16. $y = e^{x-1} + 2$

17. $E(x) = e^{x+2} - 3$

18. $y = e^{-2x}$

19. $y = e^{-x} - 1$

20. $h(x) = e^{-x} + 2$

21. Describe the graph of $y = ex + e$.

Review Problems

22. By making a sketch of the function $y = \frac{2}{x}$, calculate:

a. $\lim_{x \to \infty} y$ b. $\lim_{x \to -\infty} y$

23. a. Define e as it applies to continuous compounding in the bank.
b. Define e as a limit.
c. Is e a real number?
d. Is e rational or irrational?
e. Explain exactly why e is a valid base for an exponential function.
f. Is $e > \pi$ or is $e < \pi$?

24. a. Simplify: $e^0 + \frac{e^7}{e^2} + (e^3)^4$ b. Approximate: $e^3 + \sqrt[3]{e}$
c. Approximate: $(\pi - e)^2$ d. Approximate: e^{π}.

25. a. List the domain, range, intercepts, and asymptotes of $y = e^x$.

b. $\lim_{x \to \infty} e^x$ c. $\lim_{x \to -\infty} e^x$ d. $\lim_{x \to 0} e^x$

26. Let $f(x) = e^x$ and $g(x) = e^{x+\pi} + 100$. Describe the graph of g relative to the graph of f.

27. Graph: $y = e^{x-2} - 3$. Calculate the y-intercept and any asymptotes.

28. Graph: $y = e^{-x} - 3$. Calculate the y-intercept and any asymptotes.

29. Graph: $N(x) = e^{-x^2}$. Hint: If $x = 1$, then $e^{-1^2} = e^{-1} = \frac{1}{e} \approx 0.3679$.
Prove that the graph has y-axis symmetry.

30. Explain why the graph of $y = x^2 + ex + \pi$ is a parabola, and then prove that it has no x-intercepts.

31. True/False:

a. $\lim_{x \to \infty} f(x) = L$ means the same thing as "As $x \to \infty$, $f(x) \to L$."

b. If $P(x) = \frac{6}{1+x^2}$, then $\lim_{x \to 0} P(x) = 6$.

c. If you invest \$1 at 100% annual interest for one year, compounded 30 times a year, your final balance will be about \$2.6743.

d. If you invest \$1 at 100% annual interest for one year, compounded continuously, your final balance will be exactly \$$e$.

e. $e = \lim_{n \to 0} \left(1 + \frac{1}{n}\right)^n$

f. e can be represented exactly as the ratio of two integers.

g. e is a real number.

h. $e^4 e^5 = e^{20}$

i. The function $y = e^x$ is a polynomial.

j. The domain of the above function is $\mathbb{R}$.

k. The range of the above function is $(0, \infty)$.

l. Compared to the graph of $y = e^x$, the graph of $y = e^x + 2$ is two units higher.

m. The function $y = e^{-x}$ is decreasing.

n. The function $y = e^{-x}$ has a horizontal asymptote.

o. The function $y = e^{-x}$ has a vertical asymptote.

p. The slope of the line $y = \pi x + e$ is e.

q. The y-intercept of the graph of

$$h(x) = e^{1-x} + 2$$

is $(0, e + 2)$.

r. $e > \pi$

Solutions

1. In this limit, x is approaching 3. This means that x is approaching 3 by taking values like 3.2, 3.1, 3.05, 3.001, 3.000001 – or perhaps x is taking values like 2.9, 2.95. 2.99, 2.999, etc. Now, as x does this, what does the functional value x^2 do? Well, it gets closer and closer to 9. Therefore, the limit is 9.

2. Since x is approaching infinity, this limit is equivalent to finding a horizontal asymptote. Take your calculator and evaluate the function for a really huge value of x. You should see that the limit is 2.

3. As x shrinks toward 0, the x^2 term in the denominator essentially disappears, leaving a functional value of 9, which is the answer to the limit question.

4. a. \$2.00 b. \$2.6130 c. \$2.7146 d. \$2.7169 e. \$2.7183

5. $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$

If you invest \$1 at a 100% interest rate compounded continuously, your balance at the end of one year will be \$$e$.

6. e is an infinite, non-repeating decimal.

7. a. My TI-30X gives the answer 2.718288827, but your calculator might give 2.718281693, or something close.

b. 2.718281828 c. The second value

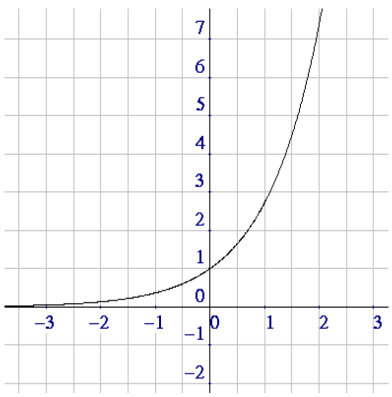
8. a. 40.4473 b. 0.3679 c. 1.1536 d. 2.5162

9. Since $1+\frac{1}{n} = \frac{n}{n}+\frac{1}{n} = \frac{n+1}{n}$, the definitions are equivalent.

10. a. 2^{x+y} b. 1 c. $\frac{1}{e^5}$ d. e^3 e. e^{x-y}

f. e^{ex} g. $10^x e^x$ h. $\frac{e^n}{2^n}$ i. $e^{10\pi x}$ j. 8

11.



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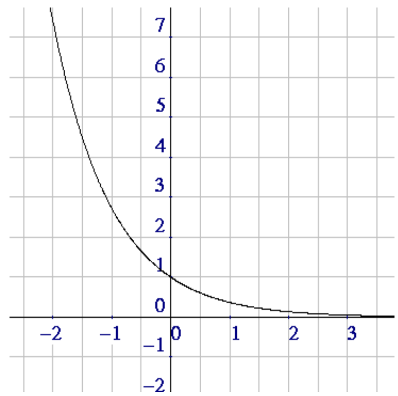
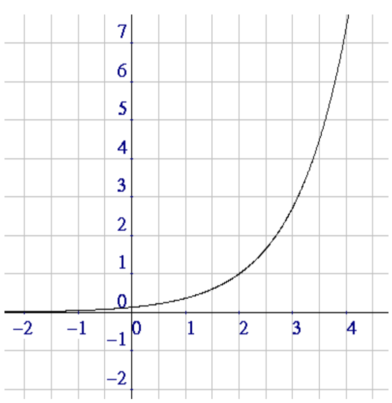


13. I'd like to see what you think the graph is.

14.



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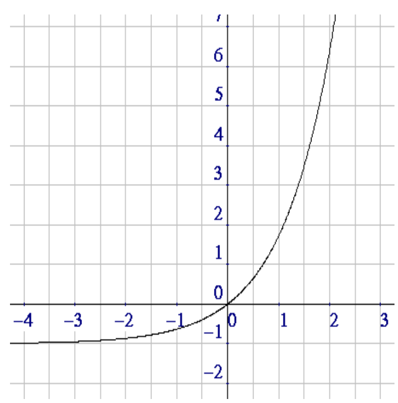

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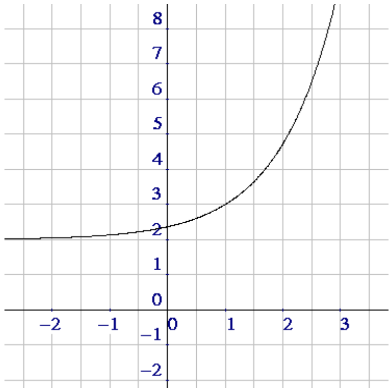

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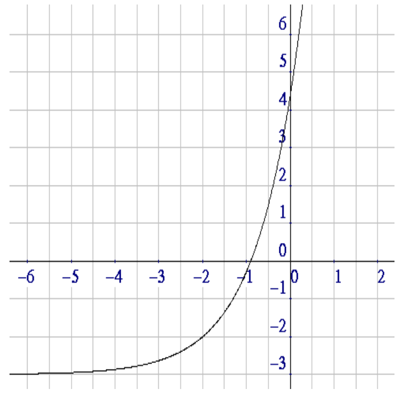

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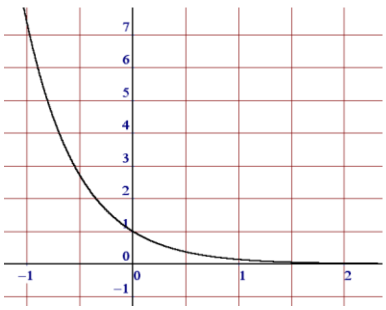

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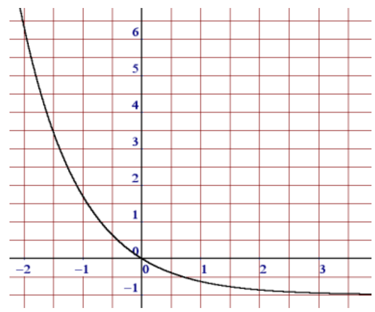

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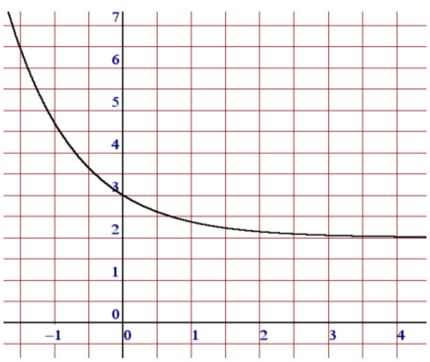


21. The graph of $y = ex + e$ is a straight line with slope e and having a y-intercept of $(0, e)$.

22. a. 0 b. 0

23. a. Invest \$1 for 1 year at an interest rate of 100%/yr compounded <u>continuously</u>. At the end of the year, the account balance is \$$e$.

b. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ c. Yes   d. Irrational

e. e is real number that is greater than 0 but not equal to 1.

f. $e < \pi$

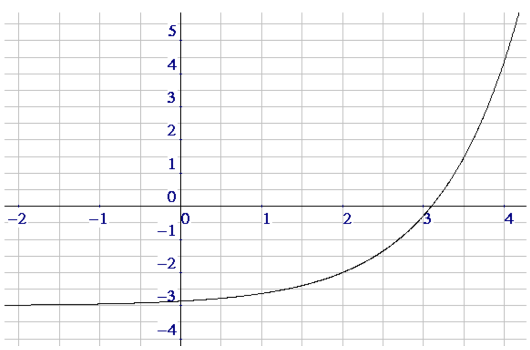
24. a. $1 + e^5 + e^{12}$ b. 21.4811   c. 0.1792   d. 23.1407

25. a. Domain = $\mathbb{R}$; Range = $(0, \infty)$;
x-int: none; y-int: (0, 1);
vert asy: none; horiz asy: $y = 0$

b. ∞ c. 0   d. 1

26. The graph of g is the graph of f shifted π units to the left and 100 units up.

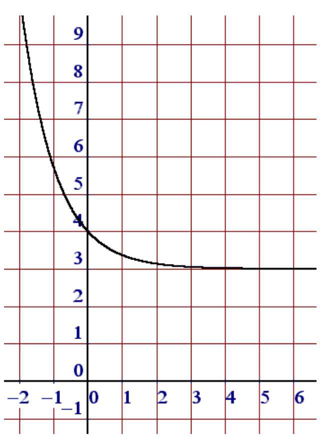
27.



y-int: $(0, e^{-2} - 3) \approx (0, -2.86)$

horiz asy: $y = -3$

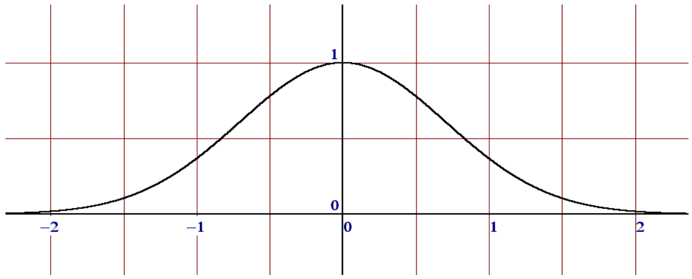
28.



y-int: (0, 4)

horiz asy: $y = 3$

29.



To prove y-axis symmetry, we change each x in the formula to $-x$ and see what we get: $e^{-(-x)^2} = e^{-x^2}$, the same formula. We're done.

30. It's a parabola because it fits the form $y = ax^2 + bx + c$; in fact, $a = 1$, $b = e$, and $c = \pi$.
To find x-intercepts, set y to 0; solve the resulting quadratic equation for x using the Quadratic Formula. You should get a negative radicand, indicating no solutions in $\mathbb{R}$, and thus no x-intercepts.

31.

a. T	b. T	c. T	d. T	e. F	f. F	g. T	h. F	i. F
j. T	k. T	l. T	m. T	n. T	o. F	p. F	q. T	r. F

"Many of life's failures are people
who did not realize
how close they were to success
when they gave up."

– Thomas Edison